SUBBUNDLES OF MAXIMAL DEGREE

MONTSERRAT TEIXIDOR I BIGAS

1. Introduction

Let C be a curve of genus g and E a generic (semistable) vector bundle of rank r and degree d. Fix a rank $r' < r$ and a degree d' for subsheaves E' of E. If $r'd - rd' = r'(r-r')(g-1)$, the number of such subbundles is finite. We shall denote this number with $m(r,d,r',g)$.

The number $m(2,d,1,g)$ had been known for a while (cf. [S, L, G]. In the case $r' = 1$, the number was computed recently by Oxbury [O] and Okonek-Teleman [OT]. A method for computing these numbers when (r', d') are relatively prime has been presented by Lange-Newstead in [LN]. A different method without any restrictions on r, d, r' is given in [Ho].

The purpose of this paper is to introduce another approach to the problem. In the present form it works only for $r' = 1$ and for $r' = 2, r = 4$. The advantage of the method is that it requires very little technical background and gives very explicit results.

We need to prove the following

1.1. Theorem *(Oxbury) Let C be a curve of genus g. Let E be a generic vector bundle on C of rank r and degree d. Choose d' so that $d - rd' = (r-1)(g-1)$. Then, the number of line subbundles of degree d' of E is r^g.*

1.2. Theorem *Let C be a curve of genus g. Let E be a generic vector bundle on C of rank 4 and degree d. Choose d' so that $2d - 4d' = 4(g-1)$. Denote by a_g(resp b_g) the number of subbundles of rank two and degree d' of E for d' even (resp odd). Then*

$$a_g = \binom{g}{0}6^g + \binom{g}{2}6^{g-2}2^2 + ... + \binom{g}{g-\epsilon}6^{\epsilon}2^{g-\epsilon}$$

$$b_g = \binom{g}{1}6^{g-1}2 + \binom{g}{3}6^{g-3}2^3 + ... + \binom{g}{g-1+\epsilon}6^{1-\epsilon}2^{g-1+\epsilon}$$

Here $g \equiv \epsilon(2),\ \epsilon \in \{0,1\}$

The author is a member of the research group "Vector Bundles on Algebraic Curves".

2. Proof of the results

The proof of the two theorems above will be done by induction on the genus. We assume the result for a curve of genus g. We then consider a curve of genus $g+1$ obtained by choosing a curve C_1 of genus one and a point P on C_1 and a curve C_g of genus g and a point Q on C_g. Glue then P and Q to obtain a curve C_{g+1} of genus $g+1$. Take a generic vector bundle E_{g+1} on C_{g+1} and count the number of subbundles of maximal rank on C_{g+1}. We then check that each of these subbundles corresponds to a non-singular point of the quotient scheme of E of suitable rank and degree, hence it should be counted with multiplicity one. This will complete the proof.

Proof of 1.1 We first check the result when $g=1$. In our situation ($r'=1, g=1$), the numerical condition for the existence of a finite number of subbundles is $d=rd'$. Then, the generic vector bundle E_1 of rank r and degree rd' on an elliptic curve is the direct sum of r generic (and therefore different) line subbundles of degree d' on C (see [T1] , Step 3 p.347). A line subbundle of degree d' of E_1 must be one of the r that appears in the direct sum decomposition. There are r of them and this agrees with the formula above.

Assume now the result for g and check it for $g+1$ using the reducible curve described above. Choose numbers r, d, d' such that

$$(*)\ d-rd'=(r-1)((g+1)-1).$$

Take then a generic vector bundle E_1 of rank r and degree $r-1$ on C_1. Take a generic vector bundle E_g on C_g of rank r and degree $d-r+1$ and glue them by a generic gluing. This gives a generic vector bundle on C_{g+1} (again by [T1]).

From (*), $(d-r+1)-rd'=(r-1)(g-1)$. Hence (by the genericity of E_g), the largest degree of a line subbundle of E_g is d'. By the semistability of E_1, the largest possible degree of a line subbundle of E_1 is zero. Hence, the only way to obtain a line subbundle of E_{g+1} of degree d' is by gluing a line subbundle of degree zero of E_1 with a line subbundle of degree d' of E_g. By induction assumption, there are r^g line subbundles of degree d' of E_g. We need to compute how many of the line subbundles of degree zero of E_1 glue with one given direction V_1 in the fiber $(E_1)_P$ of E_1 at P. Consider the exact sequence

$$0 \to E_1' \to E_1 \to (E_1)_P/V_1 \to 0$$

A subbundle of E_1 that glues with the fixed direction V_1 gives rise to a subbundle of E'_1. As $deg(E'_1) = degE_1 - (r-1) = 0$, there are r such line subbundles. Hence, E_{g+1} has $r^g \times r = r^{g+1}$ subbundles, as claimed.

We need to check now that each of them needs to be counted with multiplicity one. Equivalently, we need to show that for each such sublinebundle L_{g+1} of E_{g+1} the quotient E_{g+1}/L_{g+1} is a non-singular point of the scheme of quotients of rank $r-1$ and degree $d-d'$ of E. As the set of such L_{g+1} is finite, the dimension of the quotient scheme is zero. The tangent space to the quotient scheme at the point E_{g+1}/L_{g+1} is given by $Hom(L_{g+1}, E_{g+1}/L_{g+1}) = H^0(L^*_{g+1} \otimes E_{g+1}/L_{g+1})$. We need to check that this vector space is zero-dimensional.

From [RT] Claim p.495, the pair $L_g, E_g/L_g$ is generic in the product of the moduli spaces of vector bundles of rank one and $r-1$. Then, from Hirschowitz's Theorem ([Hi] 4.6 or [RT] 1.2), $h^0(L^*_g \otimes E_g/L_g) = 0$. Write $F_1 = E_1/L_1$. Then, a section of $Hom(L_{g+1}, E_{g+1}/L_{g+1})$ is a section of $Hom(L_1, F_1))$ that vanishes at P. Equivalently, this is a section of $L^*_1 \otimes F_1(-P)$. We need to show that $h^0(F_1 \otimes L^*_1(-P)) = 0$. This could be seen using the results in [T2], Lemma 2.5. We provide instead an ad hoc proof. From the genericity of all the data, it suffices to show that $h^0(F_1 \otimes L^*_1(-P)) = 0$ for at least one choice of data. Take as F_1 a direct sum of $r-1$ generic line bundles of degree one. Take as E_1 a generic extension

$$0 \to L_1 \to E_1 \to F_1 \to 0$$

We claim that E_1 is indecomposable. This is equivalent to showing that it is semistable. If this were not the case, there would be a subsheaf E' of E_1 contradicting semistability. We can assume that E' is semistable, otherwise it sufices to replace it with a direct summand of maximum slope. Then,

$$\frac{d_{E'}}{r_{E'}} > \frac{d_E}{r_E} = \frac{r-1}{r} = 1 - \frac{1}{r}$$

Hence, $d_{E'} > r_{E'} - r_{E'}/r$. As $r_{E'} < r$, this implies $d_{E'} \geq r_{E'}$. Consider then the exact diagram

$$\begin{array}{ccccccccc} 0 & \to & L_1 & \to & E_1 & \to & F_1 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & L' & \to & E' & \to & F' & \to & 0 \end{array}$$

As L' is a subsheaf of L_1 and F' a subsheaf of F_1 and both L_1 and F_1 are semistable, the condition $\mu(E') \geq 1$ implies $L' = 0$ and $E' = F'$ is a direct summand of F_1. But this contradicts the genericity of the extension defining E_1.

Now $F_1 \otimes (L_1)^*$ is again a direct sum of generic line bundles of degree one. Then a generic choice of P gives $h^0(F_1 \otimes L_1^*(-P)) = 0$ as required. Note that the whole picture fits together when we take as V any subspace of E_P containing $(L_1)_P$

This concludes the proof of 1.1.

Proof of 1.2. When $g = 1$, if $2d - 4d' = 4(g-1) = 0,\ d = 2d'$. For odd d', a generic vector bundle of rank four and degree $2d'$ is the direct sum of two indecomposable vector bundles of rank two and degree d' ([T1] Step3 p.347). Hence, $b_1 = 2$. If d' is even ($d' = 2\bar{d}$), then the generic vector bundle of rank four and degree $4\bar{d}$ is the direct sum of four line bundles of degree $\bar{d}$. The subbundles of rank four and degree $2\bar{d}$ are the direct sum of two of these line bundles. There are $\binom{4}{2} = 6$ such choices. Hence, $a_1 = 6$.

Assume now the result for g and prove it for $g+1$. Choose d, d' such that

$$(*) 2d - 4d' = 4((g+1) - 1)$$

Consider the same type of reducible curve of genus $g+1$ as before. Take E_1 a generic vector bundle of rank four and degree two on C_1. Take E_g a generic vector bundle of rank four and degree $d-2$ on C_g.

By the semistability of E_1, the maximum degree of a subbundle of rank two of E_1 is one. From (*), $2(d-2) - 4d' = 4(g-1)$. Hence, the maximum degree of a subbundle of rank two of E_g is d'. So, a subbundle of rank two and degree d' of E_{g+1} has its degree split as either $0, d'$ or $1, d'-1$. We point out that there are no subbundles with degree split as $1, d'$. This follows from the fact that E_1 has only a finite number of subbundles of degree one and E_g has only a finite number of subbundles of degree d'. As the gluing is generic, they cannot glue with each other.

As E_1 has rank four and degree two, it has $b_1 = 2$ subbundles of degree one and rank two. We need to know how many of the subbundles of rank two and degree $d'-1$ of E_g glue with a given subspace of dimension two V_2 of E_P. Consider the exact sequence

$$0 \to E'_g \to E_g \to (E_g)_P / V_2 \to 0$$

We see that we need to consider subbundles of rank two and degree one of E'_g. From the genericity of the gluing at P, the spaces V_2 give rise to generic vector spaces of $(E_g)_P$.From 2.1 below, E'_g is a generic vector bundle.

Note that $d-2-2 = d-4$ and from (*) $2(d-4) - 4(d'-1) = 4(g-1)$. As E'_g is a generic vector bundle of rank four and degree $d-2$, the

number of its subbundles of rank two and (maximal) degree $d'-1$ is b_g if d' is even and a_g if d' is odd. Hence, the contribution of the subbundles with splitting type $1, d'-1$ to a_{g+1} (resp b_{g+1}) is $2b_g$ (resp $2a_g$).

Look now at the splitting of the degrees as $0, d'$. We need to consider subbundles of E_1 that glue with a given subspace of dimension two V_2 of $(E_1)_P$. This is equivalent to considering subbundles of rank two and degree zero of E_1' with E_1' defined by the exact sequence

$$0 \to E_1' \to E_1 \to (E_1)_P/V_2 \to 0.$$

From 2.1 below, E_1' is generic. From the genus one case, there are $a_1 = 6$ such subbundles. We then obtain

$$a_{g+1} = 6a_g + 2b_g, b_{g+1} = 6b_g + 2a_g$$

Using this and the values of a_1, b_1, one can check the validity of the expression in 1.2 by induction on g.

The proof that these correspond to non-singular points of the quotient scheme is essentially the same as before. We only need to check that the pair consisting of such a subbundle F of degree $d'-1$ and the quotient E_g/F are generic. As F is a subbundle of maximal degree of E_g', the pair $(F, E_g'/F)$ is generic. Then, from the diagram

$$\begin{array}{ccccccccc}
0 & \to & F & \to & E_g' & \to & E_g'/F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & E_g & \to & E_g/F & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (E_g)_P/V_2 & \to & (E_g)_P/V_2 & &
\end{array}$$

Dualizing the last column, we obtain

$$0 \to (E_g/F)^* \to (E_g'/F)^* \to W_2 \to 0$$

where W_2 is a skyscraper sheaf with support on P and fiber of dimension two. Then, the genericity of E_g/F follows from 2.1.

2.1. Proposition *Let C be a curve of genus g, E a generic vector bundle of rank r and degree d. Choose any point P on C and a generic surjective map $E_P \to V_k$ where V_k is a k-dimensional vector space. Then, the kernel of the composition of the natural morphism $E \to E_P$ with the map above is a generic vector bundle of rank r and degree $d-k$.*

Proof Consider the set X of pairs consisting of a vector bundle E and a not necessarily surjective map as above. Then, X is irreducible of dimension $r^2(g-1)+1+rk$. When we require the map to be surjective, we obtain a non-empty open set in X which is therefore of the same

dimension and irreducible. To every element in X, we can associate the kernel of the composition map $E \to E_P \to V_k$. This is a vector bundle E'. Moreover such an E' appears as kernel in an rk dimensional family of these vector bundles. In fact, from

$$0 \to E' \to E \to V_k \to 0$$

one gets

$$0 \to E^* \to (E')^* \to V_k \to 0$$

and E and the surjective map can be recovered in this way. Hence, E' moves in an $r^2(g-1)+1$ dimensional set and is therefore generic. This concludes the proof of the Proposition.

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Mathematics Department, Tufts University, Medford MA 02155, USA

E-mail address: montserrat.teixidoribigas@tufts.edu